PERTH MODERN SCHOOL
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INDEPENDENT PUBLIC SCHOOL

Mathematics Specialist
Unit 3

TEST 3

Student name: Solutions **Teacher name:** ____________________

Time allowed for this task: ***45 minutes***, in class, under test conditions
Calculator-Assumed

Materials required:

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters, SCSA Formula Sheet. Classpad Calculator and Scientific Calculator.

Special items: Drawing instruments, templates

Marks available: ***44 marks***

Task weighting: **8%**

Question 1 **(7 marks)**

The points A and B have position vectors $3\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{i} + 3\underset{\sim}{k} - 2\underset{\sim}{k}$ respectively.

(a) Determine a vector equation for the straight line passing through A and B (2 marks)

$$\overrightarrow{AB} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} - (3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$
$$= -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$$

✓. ✓.

$$\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \lambda(-2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})$$

(b) Write your answer to (a) in its parametric equivalent and hence, or otherwise, express the Cartesian equation of the line in the form $\frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r}$. (3 marks)

$x = 3 - 2\lambda$ ✓.

$y = -2 + 5\lambda$ ✓.

$z = 2 - 4\lambda$ ✓.

$\frac{1}{2}$ mark each

f.t. marking

$$\lambda = \frac{x-3}{-2} = \frac{y+2}{-5} = \frac{z-2}{4}$$

✓. ✓. ✓.

(c) Determine a unit vector parallel to the straight line in (a). (2 marks)

✓.

$$|\overrightarrow{AB}| = \sqrt{(-2)^2 + 5^2 + (-4)^2} = \sqrt{45} = 3\sqrt{5}$$

$$\hat{\mathbf{r}} = \frac{1}{3\sqrt{5}}(-2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})$$

$$= \frac{\sqrt{5}}{15}(-2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})$$

✓.

-1 for not rationalising the denominator

Question 2 **(9 marks)**

A plane Π contains the two lines $r = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} + \lambda(2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k})$ and

$$r = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} + \mu(-\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k})$$

(a) Write down a vector equation of the plane Π. (1 mark)

✓

$$\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) + \mu(-\mathbf{i} + \mathbf{j} + 3\mathbf{k})$$

(b) The point $8\underset{\sim}{i} + 2\underset{\sim}{j} + c\underset{\sim}{k}$ lies in the plane Π. Determine the value of the constant c. (3 marks)

Equating **i** and **j** coefficients:

$$1 + 2\lambda - \mu = 8$$
$$-1 + 3\lambda + \mu = 2$$ ✓

$$\lambda = 2, \ \mu = -3$$ ✓

✓

$$c = 2 + 2(-1) - 3(3) = -9$$

(c) The vector $a\underset{\sim}{i} + b\underset{\sim}{j} + \underset{\sim}{k}$ is perpendicular to the plane Π. Determine the values of the constants ***a*** and ***b***. (3 marks)

✓

$$\begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = 0 \Rightarrow 2a + 3b - 1 = 0$$

✓

$$\begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = 0 \Rightarrow -a + b + 3 = 0$$

✓

$$a = 2, \ b = -1$$

(d) State the equation of the plane Π in the form $\boldsymbol{r} \cdot \boldsymbol{n} = k$. (2 marks)

✓

$$\mathbf{r} \bullet (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \bullet (2\mathbf{i} - \mathbf{j} + \mathbf{k})$$

✓

$$\mathbf{r} \bullet (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 5$$

Question 3. **(5 marks)**

(a)

(i) Find the Cartesian equation of a sphere with centre (1, -2, 3) and radius 5. (2marks)

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

$$\therefore\ (x-1)^2 + (y+2)^2 + (z-3)^2 = 5^2$$ ✓ ✓

(ii) Hence write the vector equation of this sphere. (1mark)

$$\left| r - \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \right| = 5$$ ✓

(b) Find the radius and centre of a sphere with the equation: (2marks)

$$x^2 + y^2 + z^2 - 6x + 8y + 4z + 4 = 0$$

$$x^2 + y^2 + z^2 - 6x + 8y + 4z = -4$$

$$\left(x^2 - 6x + 9\right) + \left(y^2 + 8y + 16\right) + \left(z^2 + 4z + 4\right) = -4 + 9 + 16 + 4$$ ✓

$$(x-3)^2 + (y+4)^2 + (z+2)^2 = 5^2$$

$\therefore$ The centre is $\begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$ and the radius is 5

✓

Question 4 **(9 marks)**

A particle P, begins from a point $10\underset{\sim}{j}$ m and continues with constant velocity $2\underset{\sim}{i} - \underset{\sim}{j}$ ms^{-1}. One second later another particle, starts at the point $2\underset{\sim}{i} + 15\underset{\sim}{j}$ m and moves with constant velocity $2\underset{\sim}{i} - 5\underset{\sim}{j}$ ms^{-1}.

(a) Show that the particles collide. (5 marks)

Let *t* seconds be the time after *P* starts.

$$\underset{\sim}{v}_P = 2\underset{\sim}{i} - \underset{\sim}{j}$$

Integrating $r_P = 2ti - tj + c_1$

when $t = 0$ $10j = 2 \times 0i - 0j + c_1$

$c_1 = 10\underset{\sim}{j}$ ✓ ½ mark

$$\mathbf{r_P} = 2t\underset{\sim}{i} - t\underset{\sim}{j} + 10j$$

$r_P = 2t\underset{\sim}{i} + (10-t)\underset{\sim}{j}$ ✓

$$\underset{\sim}{v}_Q = 2\underset{\sim}{i} - 5\underset{\sim}{j}$$

In∃grating $r_Q = 2ti - 5tj + c_2$

when $t = 1$ $2i + 15j = 2 \times 1i - 5 \times 1j + c_2$

$c_2 = 20\underset{\sim}{j}$ ✓ ½ mark

$\mathbf{r_Q} = 2t\underset{\sim}{i} - 5t\underset{\sim}{j} + 20\underset{\sim}{j}$
$= 2t\underset{\sim}{i} + (20 - 5t)\underset{\sim}{j}$ } ✓

For collision to occur $\mathbf{r_P = r_Q}$

$2ti + (10-t)j = 2ti + (20-5t)j$ ✓

Equating Components

$2t\,i = 2t\,i$ and $(10 - t)j = (20 - 5t)j$ the i component is ok

$10 - t = 20 - 5t$

$t = 2.5$ The particles collide 2.5 sec after P starts "moving

✓

(b) Find the Cartesian equations of their paths. (2 marks)

Looking a the Components of r_P

$x = 2t$ and $y = 10 - t$ (Eliminating t gives)

$$y = 10 - \frac{x}{2}$$ ✓

Looking at the Components of r_Q

$x = 2t$ and $y = 20 - 2t$ (Eliminating t gives)

$$y = 20 - \frac{5x}{2}$$ ✓

(c) Find the Cartesian coordinates for the point of collision (2 marks)

Solving simultaneously we get

$$10 - \frac{x}{2} = 20 - \frac{5x}{2}$$ ✓

$$x = 5$$

Substituting $x = 5$

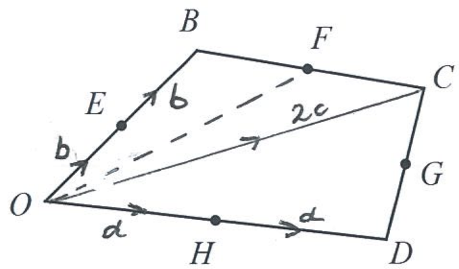
$$5 \times (5) + 2y = 40$$

$$\therefore y = 7 \cdot 5$$

The particles collide at the point **(5 , 7·5)** ✓

Question 5 **(7 marks)**

In the diagram below, E, F, G and H are midpoints of the sides of the quadrilateral $OBCD$.



Let $\overrightarrow{OB} = 2\mathbf{b}$ $\overrightarrow{OC} = 2\mathbf{c}$ and $\overrightarrow{OD} = 2\mathbf{d}$.

(a) Show that $\overrightarrow{OF} = \mathbf{b} + \mathbf{c}$. (2 marks)

$$\overrightarrow{OF} = \overrightarrow{OB} + \tfrac{1}{2}\overrightarrow{BC} \quad ✓$$
$$= \overrightarrow{OB} + \tfrac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OB}\right)$$
$$= 2\mathbf{b} + \tfrac{1}{2}(2\mathbf{c} - 2\mathbf{b})$$
$$= \mathbf{b} + \mathbf{c} \quad ✓$$

(b) Determine $\overrightarrow{OG}$ in terms of ***b, c, and d*** (2 marks)

$$\overrightarrow{OG} = \overrightarrow{OD} + \tfrac{1}{2}\overrightarrow{DC} \quad ✓$$
$$= \overrightarrow{OD} + \tfrac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OD}\right)$$
$$= 2\mathbf{d} + \tfrac{1}{2}(2\mathbf{c} - 2\mathbf{d})$$
$$= \mathbf{d} + \mathbf{c} \quad ✓$$

(c) Prove that $EFGH$ is a parallelogram. (3 marks)

$$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE}$$
$$= \mathbf{b} + \mathbf{c} - \mathbf{b} \quad ✓$$
$$= \mathbf{c}$$

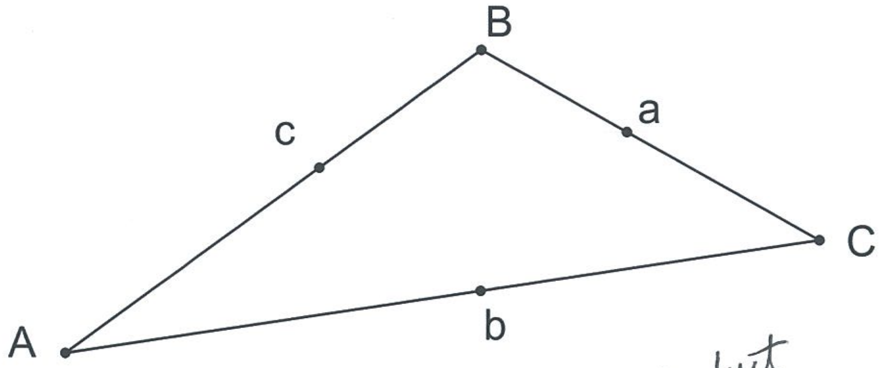
$$\overrightarrow{HG} = \overrightarrow{OG} - \overrightarrow{OH}$$
$$= \mathbf{d} + \mathbf{c} - \mathbf{d}$$
$$= \mathbf{c} \quad ✓$$

✓ must have both

Hence EF is parallel to HG and of equal length and so $EFGH$ must be a parallelogram.

Question 6 **(7 marks)**

Use the vector product (cross product) to find the area of the triangle with vertices A(-1,3,2), B(3,5,1) and C(1,6,-2)



Recognises "Cross Product"

Area = $\frac{1}{2}ab\sin(C)$ ✓

= $\frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{AC}|$

$\overrightarrow{BC} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$ ✓ $\overrightarrow{AC} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$ ✓

Vector Cross Product (Classpad): $\overrightarrow{BC} \times \overrightarrow{AC} = 5i - 14j - 4k$ ✓

Finding the Norm. (Classpad) $|\overrightarrow{BC} \times \overrightarrow{AC}| = \sqrt{285}$ ✓

∴ **Area =** $\frac{1}{2}\sqrt{285}$ ✓

≈ **8·44 sq units** ✓